Assignment 6

Astronomy 541

Assignment: Due Wednesday, March 11, in class

These problems have a lot of text because we were trying to make them self-contained and to discuss some of the context. We were worried that the actual tasks to be performed would get lost in all the words, so we've underlined the key words to guide the eye.

Problem 1 (8 pts): In class, we said that the scale of the acoustic peaks in the CMB power spectrum was set by the comoving distance that a sound wave could travel between the time when the perturbation was created ($t \approx 0$) and the epoch of recombination ($t = t_*$). In other words,

$$s = \int_0^{t_*} dt\; c_s(z)\;(1+z)$$

where c_s is the sound speed. s is called the sound horizon. In this problem, we will first calculate s and then compute the dependence of the angular scale of the acoustic peaks on cosmological parameters.

Throughout the problem, you should quote your answers in terms of the baryon density Ω_b and the total matter density (baryons plus dark matter) Ω_m as well as Ω_Λ, $\Omega_K = 1 - \Omega_m - \Omega_\Lambda$, and $h = H_0/(100\ \text{km s}^{-1}\ \text{Mpc}^{-1})$. Assume that the CMB temperature is 2.725 K today.

a) The sound speed results from the competition between restoring forces and inertia. For a simple fluid, $c_s^2 = dp/d\rho$. Before recombination, the photons and baryons can be assumed to be locked together (for perturbations of long wavelength). For the photons, $p_\gamma = \rho_\gamma c^2/3$, but for the baryons $p_b \approx 0$.

Compute the sound speed of the combined fluid as a function of redshift by considering the response of the pressure and density to an adiabatic (no heat conduction) compression by a linear factor a. This means that $c_s^2 = (dp/da)/(d\rho/da)$. Remember that the photons have their pressure and density scale by a^{-4}, while the baryon density scales only as a^{-3}.

Show that for $\Omega_b h^2 = 0.02$, the approximation that $c_s \approx c/\sqrt{3}$ is good to $\sim 30\%$ for $z > 1000$.

For the rest of the problem, assume $c_s = c/\sqrt{3}$. The approximation isn't necessary, but we want to simplify the integral in (c).

b) Our calculation of s will involve $H(z)$ at early times. As usual, $H(z)$ can be calculated from the Friedmann equation. At early times, curvature and cosmological terms are negligible, so

$$H(z) = H_0 \left[\Omega_r (1+z)^4 + \Omega_m (1+z)^3\right]^{1/2}$$

where Ω_r is the energy density of relativistic particles today.

Remembering that the neutrinos contribute an extra component of relativistic density that adds $3(7/8)(4/11)^{4/3} = 0.68$ to that of the photons, compute the redshift z_{eq} at which the density of relativistic species and matter are equal.

c) Now compute the sound horizon s. Assume recombination occurs at redshift z_*, which we'll take to be 1000. Argue why one cannot neglect the radiation contribution to $H(z)$ in this calculation. Hint: you've done this problem once before this semester.

In truth, recombination depends slightly on the baryon density $\Omega_b h^2$ and the matter density $\Omega_m h^2$, but this can be modelled accurately.

d) The characteristic angular scale in the CMB will be $s/S[r(z_*)]$, where $S[r(z)]$ is our familiar metric distance (see assignment 2). You might have expected the angular diameter distance $D_A = S[r(z)]/(1+z)$ to appear here, but that's the angular diameter of an object of a given physical (non-comoving) size. Since our sound horizon was defined as a comoving distance, it's physical size is $s/(1+z)$. That distance divided by D_A is just $s/S[r(z_*)]$.

Since we typically study the CMB in terms of power spectra, we will invert this angular scale into a typical angular wavenumber (or multipole) $\ell_{acoustic} = 4S[r(z_*)]/s$. The constant factor of 4 is picked to match $\ell_{acoustic}$ to the first acoustic peak.

Whereas the value of s was essentially independent of the value of Ω_Λ and Ω_K, $S[r(z_*)]$ does depend quite a bit on these. You might be concerned that it would *also* depend on the radiation density because $r(z_*)$ formally depends on $H(z)$ at z_*, where the radiation density is not negligible. Argue that for $z_{eq} > z_*$ it is a good approximation to ignore the radiation term in $H(z)$ when computing $r(z_*)$.

e) Compute $\ell_{acoustic}$ for a fiducial universe of $\Omega_m = 1$, $h = 0.5$. Show that the results are insensitive to (though not strictly independent of) the choice of h (e.g. try $h = 0.3$ and $h = 0.7$).

f) Compute $\ell_{acoustic}$ for a $\Lambda = 0$ universe with $\Omega_m = 0.2$, 0.4, and 1. Use $h = 0.7$. You can reuse your work from assignment 3, either using the analytic formula for $S(z)$ or your numerical integration.

g) Now compute $\ell_{acoustic}$ for a flat universe with non-zero Λ. Use $\Omega_m = 0.2$ and 0.4 with $h = 0.7$. Computing $r(z)$ in this cosmology requires a numerical integration. You only need modest accuracy (e.g. 1%). You've already done this integral in assignment 3.

From parts (e), (f), and (g), you should have demonstated that the position of the acoustic peaks depends primarily on Ω_K and only slighly on Λ and h! This is why people claim that the CMB anisotropy observations suggest a flat universe!

Problem 2 (7 pts): a) Calculate the mean free path length of a photon in the early universe as a function of redshift, assuming that the dominant opacity is Thompson scattering and that all of the electrons in the universe are ionized. Assume $\Omega_b h^2 = 0.02$, $\Omega_m h^2 = 0.14$, and treat all the baryons as hydrogen. Compare your result to the Hubble distance (i.e. to $c/H(z)$).

b) At $z = 1000$, what ionization fraction would be needed to allow the mean free path to be equal to the Hubble distance (which is the rough criteria for the photons to stream freely past the elecrons).

That this number is less than 1 means that the recombination of the electrons and protons is important to the physics of the CMB. In particular, because recombination sweeps the ionization fraction from 1 to about 10^{-4} in about 10% of the Hubble time (then), it means that the photons we see last scattered in a rather thin shell in redshift.

Alternative cosmologies might keep the hydrogen ionized even at $z < 1000$ (by some large amount of energy injection, of course). In such cosmologies, the photons still eventually decouple from the electrons (you can use part (a) to say when this is), but they do so over an entire Hubble time. The resulting last-scattering surface is very thick, resulting in a significant weakening of the CMB anisotropies.

c) Prior to recombination, the mean free path is fairly short compared to the Hubble time. The photons will therefore execute a random walk in the sea of electrons. If the universe were homogeneous, then this motion would be irrelevant; photons leaving one area would be replaced by new ones arriving. However, the universe does have small perturbations, and the effect of the random walk of the photons is to smooth out the hot spots and fill in the cold spots. In other words, we have a diffusive damping of the perturbations (much as diffusive effects damp out short wavelength sound waves). In cosmology, this process is called Silk damping.

Compute the comoving distance that a photon can random walk between the Big Bang and $z = 1000$. Assume that the mean free time much shorter than the Hubble time (as found in part (a)) and that each scattering isotropizes the photon's momentum. Give your answer in Mpc, adopting the cosmology from above.

Hint: consider a time interval much longer than the mean free time but sufficiently shorter than the Hubble time that the mean free time can be considered constant. The root mean square of the physical distance travelled is then the mean free path times the square root of the number of mean free times in the time interval. You can then sum up many such time intervals (as an integral) by adding these individual walks in quadrature (i.e. add the squares and take the square root of the sum).

In truth, this result underestimates the Silk damping scale. The problem is that the reionization of the universe isn't instantaneous, and so the universe spends a non-negligible amount of time in a regime where the mean free path is comparable to the Hubble length. Some fraction of the photons rescatter in this period, thereby allowing them to smooth over perturbations on a larger scale.

d) If you consider this length scale as a transverse distance on the sky at $z = 1000$, what angle is this in arcminutes? Use one of the cosmologies from problem 1(g). Below this angular scale, one would expect the anisotropies of the CMB to be smooth!