Assignment 5

Astronomy 541

Assignment: Due Wednesday, February 25, in class

Problem 1 (4 pts): For $\Omega_m = 1$ and $c_s = 0$, show that the gravitational potential ϕ of a growing-mode perturbation is time independent (as usual, ignore the homogeneous term in the Poisson equation).

Argue that in a low-density universe, where the growth lags behind $D \propto R$ and eventually converges to a constant, the gravitational potential decays to zero.

Problem 2 (6 pts): Consider a spherically symmetric perturbation at time t_0

$$\delta(r) = \begin{cases} \delta_0 & \text{for } r < A \\ -\delta_0/7 & \text{for } A < r < 2A \\ 0 & \text{for } r > 2A \end{cases}$$

This was chosen so that the total mass of the perturbation is zero (sometimes called a *compensated* perturbation). We assume linear perturbation theory ($\delta_0 \ll 1$), zero pressure, and $\Omega_m = 1$.

a) Compute the gravitational acceleration (neglecting any homogeneous term, of course) as a function of r.

b) What is the growing mode velocity field at t_0? For $r < A$, write the velocity in terms of the Hubble expansion rate Hr and the density perturbation δ_0.

c) In linear perturbation theory, the time evolution of the perturbation is simply an overall rescaling of the amplitude of the perturbation by a function $D(t)$ (where $D(t_0) = 1$). If the initial velocity is zero everywhere, what is $D(t)$? Derive this by decomposing the perturbation into growing and decaying modes and then adding appropriately.

d) Consider that A is $20h^{-1}$ Mpc and that $\delta_0 = 0.3$ today. What is peak of the infalling velocity?

Problem 3 (4 pts): In class, we derived the following differential equation for the evolution of the amplitude of small perturbations:

$$\frac{d^2D}{dt^2} + 2H\frac{dD}{dt} = \left(4\pi G\rho_h - \frac{c_s^2k^2}{R^2}\right)D$$

Remember that H and ρ_h (the average density of the universe) are functions of time.

a) For pressureless matter in an open universe with $\Lambda = 0$, the above equation has the solution

$$D(t) = 1 + \frac{3}{x} + \frac{3\sqrt{1+x}}{x^{3/2}} \ln\left[\sqrt{1+x} - \sqrt{x}\right],$$

where $x = R(t)(1-\Omega_m)/\Omega_m$ and $R = 1$ today. Here, Ω_m is the density of matter.

If $\Omega_m = 0.3$, by what factor has the amplitude of perturbations grown between $z = 99$ and today? Between $z = 1$ and today? Compare these results to the those in an $\Omega_m = 1$ universe.

b) For Λ cosmologies, the solution for the growth function requires special functions (elliptic functions or beta functions). However, there is a fitting formula (Carroll, Press, Turner 1992, adapted from Lahav et al 1991) that holds for matter-dominated universes with curvature and Λ. The formula says that the growth function at $z = 0$ relative to that at a large initial z is

$$\frac{D(0)}{D(z_i)(1+z_i)} = \frac{5\Omega_m}{2}\left[\Omega_m^{4/7} - \Omega_\Lambda + (1+\Omega_m/2)(1+\Omega_\Lambda/70)\right]^{-1}$$

At large z the growth function scales as $(1+z)^{-1}$, so we don't need to specify a particular z_i.

Using the formula, compute the factor by which the amplitude of structure has grown from $z = 99$ to $z = 0$ for a universe of $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Compare this to the open universe in part (a) and to the $\Omega_m = 1$ case. You might want to try checking your results from part (a) too!

For future reference, to apply this formula to get the growth at other redshifts, you have to rescale Ω_m and Ω_Λ to the value that an observer at that redshift would measure and then divide by $1 + z$ to accomplish a rescaling of z_i. If that's confusing, consider the formula for $\Omega_m = 1$ and $\Omega_\Lambda = 0$ to understand the z_i correction and then consider that the formula is essentially saying how much a cosmology's growth lags that of Einstein-de Sitter given a common beginning.

Problem 4 (6 pts): Consider the relic population of neutrinos. I argued in class that at $kT \gg 1$ MeV neutrinos (and antineutrinos) interact quickly enough that they are populated at their thermal abundances. At $kT \approx 1$ MeV, the neutrinos stop interacting with the rest of the particles. After that time, the annihilation of the electrons and positrons heats the photons to a temperature that is $(11/4)^{1/3}$ higher than the temperature of the neutrinos.

We will consider here that the neutrinos have a non-zero mass. The mass is negligible near the decoupling redshift, so one can use the ultra-relativistic limit $E = pc$. But at low redshift, we will assume the mass is large enough that the neutrinos are non-relativistic.

a) What is the velocity distribution of the massive neutrinos today? In other words, what is dn/dv, ignoring the overall normalization of the number density? To compute this, recall that the neutrinos at high temperature are in a thermal distribution for a massless fermion and that the temperature at the decoupling redshift z_d is $(4/11)^{1/3}(1+z_d)2.725$ K. After decoupling, the momenta scale as $(1+z)^{-1}$ (not energies, not velocities). You should compute the momemtum distribution at z_d and then convert is to the velocity distribution today. Note that you do not need to compute z_d; it will cancel out.

b) Compute the mean velocity of the neutrinos today. The following integrals may be useful:

$$\int dx \frac{x^n}{e^x - 1} = n!\,\zeta(n+1) \quad (1)$$

$$\int dx \frac{x^n}{e^x + 1} = n!\,\zeta(n+1)(1-2^{-n}) \quad (2)$$

where $\zeta(m) = \sum_{k=1}^{\infty} k^{-m}$ is the Riemann zeta function. $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202$, $\zeta(4) = \pi^4/90$.

c) Is this distribution the same as that of a thermal distribution of a massive non-relativistic particle (i.e., the Maxwell distribution)? If one had a Maxwell distribution with the same mass and with temperature $(4/11)^{1/3}T_{CMB}$, what would the mean velocity be? Would you really say that the neutrinos have a temperature today of $(4/11)^{1/3}T_{CMB}$?