Assignment 4

Astronomy 541

Assignment: Due Wednesday, February 18, in class

Problem 0 (not to be turned in): Verify the statements in lecture that:

a) the ratio of the number density of photons to that of nucleons is about 1.8×10^9 for $\Omega_b h^2 = 0.02$. Recall that $\rho_b = \Omega_b(3H_0^2/8\pi G)$. The number of photons in a blackbody spectrum is $20.3T^3$ cm^{-3} K^{-3}.

b) the energy density in the CMB at redshift z corresponds to about $(1+z)$ MeV per nucleon. Again, use $\Omega_b h^2 = 0.02$.

Problem 1 (5 pts): For a universe composed only of radiation ($w = 1/3$), compute the time $t(z)$ between the Big Bang and a given redshift (note that this formula will be slightly different than the formula in Assignment 2). Express this time both in terms of H_0 and redshift and in terms of the Hubble parameter at the final redshift $H(z)$.

Compute the comoving distance $r_H(z)$ travelled by a (non-interacting) light ray between the Big Bang and a given redshift (again, note that this formula is slightly different than the $r(z)$ formula you've been using). Express this in terms of z and $t(z)$ and interpret the result.

Problem 2 (5 pts): a) Compute $r_H(z)$ for a general cosmology of radiation, matter, curvature, and a cosmological constant. However, you should confine yourself to early times (high redshift) where the contribution to $H(z)$ from curvature and a cosmological constant is negligible. In other words, assume $H(z) = H_0\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3\right]^{1/2}$ but allow $\Omega_r + \Omega_m \neq 1$. Note: this integral should be done analytically, not numerically.

Demonstrate that the limits of $r_H(z)$ at early times match the behavior of Problem 1.

b) The CMB plus the predicted neutrino backgrounds make $\Omega_r h^2 = 4.2 \times 10^{-5}$. For a universe with $h = 0.7$ and $\Omega_m = 0.3$, compute the value of the r_H at $z = 1000$. This is the comoving distance that a causal signal can propagate prior to $z = 1000$ in a universe of radiation and matter. How does this size compare to the present-day size of the universe?

Repeat the calculation for the $z = z_{eq}$, where z_{eq} is the redshift at which the matter and radiation energy densities are equal.

These are examples of a *particle horizon.* Extrapolating to early times, the causally connected portion of the universe is shrinking to zero in comoving coordinates! This raises the question of how we can appeal to early causal physics to explain the homogeneity of the universe.

Problem 3 (5 pts): a) If the atoms in the universe were ionized, then the free electrons would scatter photons (at least non-gamma-rays) at the Thompson cross-section $\sigma_T = 6.65 \times 10^{-25}$ cm^2. Assume that the density of nucleons is 3.8×10^{-31} g cm^{-3} today ($\Omega_b h^2 = 0.02$) and consider that all of the baryons are in hydrogen (ignore helium). Charge neutrality insists that the number of electrons is equal to the number of protons. If the universe were fully ionized, compute the optical depth a photon encounters on its way from redshift z to us. Assume that the universe has $\Omega_0 = 1$ and ignore any radiation contribution to the Hubble constant.

Hints: Compute the density of electrons today and write down the scaling with redshift. The optical depth is then the integral along the line of sight of the cross-section times the density. You can change variables from $d\ell$ to dz. It sounds more complicated than it is! See also pages 224-225 of Longair.

b) At what redshift is the optical depth unity? Assume $H_0 = 70$ km s^{-1} Mpc^{-1} (i.e. $h = 0.7$).

Problem 4 (5 pts): Consider that we change particle physics so as to include a yet-undiscovered stable massive particle. For simplicity, we will make it spin-0 (meaning that $g = 1$) and call it X. X and its antiparticle $\bar{X}$ interact quickly enough in the early universe that their number densities reach thermal equilibrium. We will imagine that the mass m_X is large, of order the proton mass or larger.

a) If X interacts rarely enough, then it will decouple (interaction rate less than Hubble parameter) when the universe is still hotter than $m_X c^2$. In that case, X remains in a thermal distribution today. Show that this is a cosmological catastrophe by computing $\Omega_m h^2$ in terms of m_X (where the latter is measured in GeV).

b) If X interacts more quickly, then it remains able to follow the thermal equilibrium prediction as the temperature drops below the rest mass. In other words, as the temperature drops, the X and $\bar{X}$ can annihilate. However, as the number density drops, the annihilation reaction slows and freezes out. This leaves a relic population of X and $\bar{X}$ particles that might be the dark matter today.

Compute the relic abundance of X and $\bar{X}$ particles as a function of the annihilation cross-section σ_a and the mass m_X. You may assume that the reaction ends when the reaction rate is equal to the Hubble parameter. You may assume that the Hubble parameter is to be computed at a time when the temperature of the universe is $0.1 m_X c^2/k$. You may assume that $g_* = 100$; g_* is the total number of relativistic spin-states, including a penalty of 7/8 for fermions. Assume a zero chemical potential.

Hence, show that *if* the X particle is to be the dark matter ($\Omega_X h^2 \approx 0.11$), then it must have a cross-section that is predicted (at least to a factor of $\sim$3) by its mass!

This is an example of how cosmology can put limits on particle physics: we've just placed an lower limit on the cross-section of a stable particle as a function of its mass (barring heroic attempts to violate our assumptions, of course).