Assignment 1

Astronomy 541

Monday, January 19 is a holiday, so there is no class. We will have class on Wednesday, January 21.

To alert you in advance, there will not be lectures on Monday February 9 and Wednesday February 11; instead there will be lectures on Fridays February 6 and 13.

In the next few weeks, we will studying material found in chapters 5, 7, and 8 of Longair's *Galaxy Formation.*

- **Chapter 1:** A summary of cosmology, including some history, that you should read.
- **Chapter 2:** Some general ideas. We'll touch on all of these in the course, but we're not starting there. Optional.
- **Chapters 3 and 4:** Discussion of galaxies and clusters. Some of the material was covered in AST 540; the rest we will return to later in the course. Skip these for now.
- **Chapter 5:** This covers the derivation and applications of the standard geometries and coordinate systems. Don't sweat the details in section 5.2.
- **Chapter 6:** This introduces general relativity. However, the only result from GR that we need at this time is given, without derivation, as the first equations of chapter 7. You will not need chapter 6 at the level of this course; it can be skipped for now.
- **Chapter 7:** This covers all of the gravitational effects in homogenous cosmology. Sections 7.5, 7.7, and A7 can be skipped or skimmed.
- **Chapter 8:** This covers the current observational status of the classical cosmological tests. Don't sweat the details in section 8.6; we will return to this topic later.

If you want a more rigorous, GR-based derivation of this material, we would suggest sections 2.1 through 3.1 of Kolb & Turner's *The Early Universe.*

While the problems that we assign for credit will generally be more exact calculations, we want to encourage you to practice your skills at "back of the envelope" calculations. In that spirit, let us suggest that you estimate answers to the following questions and that you and a study partner come up with similar questions to practice on. Note that some of these questions don't necessarily have a single, simple answer; instead, you'll need to recognize unspecified aspects and incorporate your assumptions along the way. [These questions assume some astronomical background. If you didn't take AST 540, you may not yet have the preparation. Not to worry! Try discussing the questions with someone who has taken AST 540 so that you can learn some of the context.] Note that the following are *not* being graded; they are here for your own practice.

1) If the protons locked up in the stars in galaxies were spread evenly throughout the universe, what would the density be? What would the number density be? How does this compare to the density in the interstellar medium of the Milky Way?

2) How does the rest-mass energy density of these protons compare to the energy density of the CMB, which is a blackbody of temperature 2.725 Kelvin?

Assignment: Due Wednesday, January 28, in class

Problem 1 (5 pts): Show that the metric for a 3-sphere (i.e. a sphere in 4-dimensional space) of radius R_c is

$$d\ell^2 = dr^2 + R_c^2 \sin^2(r/R_c) \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) . \quad (1)$$

Hints: The 3-sphere can be written as the surface $w^2 + x^2 + y^2 + z^2 = R_c^2$ in 4-dimensional Cartesian space. This surface can be parameterized as

$$\begin{aligned} w &= R_c \cos\chi \\ z &= R_c \sin\chi \cos\theta \\ x &= R_c \sin\chi \sin\theta \cos\phi \\ y &= R_c \sin\chi \sin\theta \sin\phi \end{aligned}$$

The distance between two infinitesimally close points on the surface doesn't depend on whether one measures in the 4-space or in the 3-dimensional manifold. So one can start with the Cartesian metric $d\ell^2 = dw^2 + dx^2 + dy^2 + dz^2$ and substitute the differentials based on the change of coordinate systems, e.g. $dw = -R_c \sin\chi \; d\chi$. After simplifying this, change variables once more, setting $r = R_c\chi$.

Problem 2 (5 pts): The metric of the homogeneous hyperbolic 3-space is

$$d\ell^2 = dr^2 + R_c^2 \sinh^2(r/R_c) \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

for a constant R_c (note the small change from equation 1).

Show that the substitution $x = R_c \sinh(r/R_c)$ makes the metric

$$d\ell^2 = \frac{dx^2}{1 - \kappa x^2} + x^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \quad (2)$$

where $\kappa = -1/R_c^2$.

Show that a similar substitution takes the spherical metric (1) to the form (2) but with $\kappa = +1/R_c^2$.

Noting that the flat metric is simply (2) with $\kappa = 0$, you can see that all three homogeneous metrics can be written in a single form.

In words, the difference between these coordinate systems is whether we label the radial direction by the distance along the radial spoke or the circumference of the circle (divided by 2π).

Problem 3 (5 pts): The purpose of a metric is to measure distances between nearby points. In this problem, you will apply the metric to measure the length of curve.

Let us consider the 2-sphere (in 3-dimensional space) of radius R_c. The metric in spherical coordinates is

$$d\ell^2 = R_c^2 d\theta^2 + R_c^2 \sin^2\theta \, d\phi^2 \quad (3)$$

We know that the length along a great circle is $2\pi R_c$. This can be easily calculated if we rotate the spherical coordinates so that the great circle is the equator. In that case, $\theta = \pi/2$ and $d\ell = R_c\, d\phi$. Integrating over the ϕ range of 0 to 2π gives the circumference $2\pi R_c$.

For practice, let us now do the problem the hard way in which we incline the great circle relative to the equator.

a) First, remember that a great circle is the intersection of the sphere with a plane that contains the origin. Let's pick the normal to the plane to be $\hat{n} = \sin\beta\ \hat{y} + \cos\beta\ \hat{z}$. The great circle will intersect the equator along the x-axis. $\beta = 0$ will give the equator; $\beta = \pi/2$ will give a meridian. You will investigate the range in between these two limits.

For the coordinate choice

$$\begin{aligned} x &= R_c \sin\theta\ \cos\phi \\ y &= R_c \sin\theta\ \sin\phi \\ z &= R_c \cos\theta, \end{aligned}$$

use $\vec{r} \cdot \hat{n} = 0$ to show that the locus of points on the great circle is given by

$$\cot\theta = -\tan\beta \sin\phi \tag{4}$$

b) Now that you know an implicit relation for θ as a function of ϕ, use the metric (3) to express the distance along the great circle as a relation between $d\ell$ and $d\phi$.

Next, integrate $d\ell$ over ϕ from 0 to 2π to get the length of the great circle. The following integral should be helpful:

$$\int_0^{\pi/2} \frac{dx}{1 + a^2 \sin^2 x} = \frac{\pi}{2\sqrt{1 + a^2}}$$